the technique used previously by Mitchell, Crocco, and Sirignano[1] and Crocco and Mitchell.[2] The relevant dependent variables u, p, r, C, σ and w are represented as power series in ϵ. For example, $p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \ldots$ etc., where $\epsilon = M^{1/2}$. The coefficients of ϵ in these expansions are comprised of a steady-state and a time-dependent part. Thus, $p_1 = \bar{p}_1 + p_1^1$, $p_2 = \bar{p}_2 + p_2^1$ etc. Both u_l and $\bar{\eta}$ are taken to be of $0(\epsilon)$. Since, as will be discussed shortly, the analysis is carried out through $0(\epsilon^5)$, the combustion zone can actually occupy a fairly sizeable fraction of the chamber axial length, and is not of zero length as in the "concentrated combustion" model used by Mitchell, Crocco, and Sirignano.[1]

It is necessary to use the simplest kind of coordinate stretching in order to ensure well-behaved periodic solutions even when waveforms are discontinuous. Therefore, the stretched time variable δ is introduced and $P = 2(1 + \epsilon P_1 + \epsilon^2 P_2 + \ldots)$. P is the nondimensional period of the oscillations and 2 is the wave travel time for an acoustic (zero amplitude) wave. The technique of multiple scales is applied with the introduction of a second axial variable y, which is defined as $y = x/\eta$. Derivatives of $\bar{u}$ and $\bar{w}$ with respect to this variable are of $0(1)$ instead of $0(1/\epsilon)$ as they are with respect to x. All dependent variables are then considered to be functions of the 3 independent variables x, y, and δ. Thus, $u = u(x,y,\delta)$, $p = p(x,y,\delta)$, etc. The governing partial differential equations and the droplet vaporization equation are then rewritten in these variables, and the power series representations of the dependent variables are substituted into the equations which result.

The system of equations is first solved for the steady state. To the order of approximation necessary for consistent solution of the time-dependent equations, the results are

$$\bar{u}_2 = \bar{w}_2 = 1 - (1 - y)^{3/2}, \quad \bar{p}_0 = 1, \ \bar{p}_3 = \gamma \bar{u}_2 u_l$$

$$\bar{u}_1 = \bar{w}_1 = \bar{u}_3 = \bar{w}_3 = \bar{p}_1 = \bar{p}_2 = \bar{\sigma}_0 = \bar{\sigma}_1 = \sigma_2 = 0$$

$$r_0 = (1 - y)^{1/2}, \quad \bar{\eta}_1 = u_l/2C_0$$

Carrying out the analysis of the time-dependent equations through $0(\epsilon^3)$ and applying the appropriate boundary conditions leads to the following expressions:

$$u_2^1 = f(\delta - x) - f(\delta + x) \quad p_2^1 = \gamma[f(\delta - x) + f(\delta + x)]$$

$$r_2^1 = \frac{1}{2}(1 - y)^{-1/2}\left[2\int_{\delta}^{\delta - \bar{\psi} y} C_1^1 d\delta^1 - y\,\frac{\eta_2^1}{\bar{\eta}}\right]$$

$$\eta_3^1 = -2\bar{\eta}_1 \int_{\delta - \bar{\psi}}^{\delta} C_1^1 d\delta^1$$

$$u_1^1 = p_1^1 = p_3^1 = u_3^1 = \sigma_1^1 = w_1^1 = w_2^1 = w_3^1 = 0$$

Here, $\bar{\psi} = \bar{\eta}_1/u_l$ (the steady-state droplet lifetime) and f is an arbitrary function periodic in 2. In order to determine the form of f and therefore of p_2 and u_2, it is necessary to continue the analysis through $0(\epsilon^5)$. Doing this and applying the appropriate order boundary conditions, the following nonlinear integro-differential equation for f is finally derived:

$$\left[2P_2 + \sigma_2^1 + (\gamma + 1)f - \frac{3 - \gamma}{2}\langle f\rangle\right]\frac{df}{d\delta} =$$

$$(3\gamma - 1 - 2\gamma d)f + 3\gamma d\int_0^1 (1 - y)^{1/2} f(\delta - \bar{\psi} y)dy \quad (2)$$

where $\sigma_2^1 = 2(\gamma - 1)\langle f\rangle$,

$$\langle f\rangle = \frac{1}{2}\int_{\delta - 2}^{\delta} f(\xi)d\xi, \; d = \frac{\gamma - 1}{\gamma}\left[\frac{1}{2} + \frac{\bar{B}}{(1 + \bar{B})(1 - \bar{\tau})\ln(1 + \bar{B})}\right]$$

The developments that resulted in Eq. (2) are valid for either continuous or discontinuous pressure waves. In order to determine the waveform and amplitude of the pressure oscillations, this equation must be solved for either type of oscillations. Solution of Eq. (2) for small f is easily carried out by linearizing the equation and assuming $f = \sin\pi\delta$. A neutral stability relationship results. This is

$$d = \frac{3\gamma - 1}{2\gamma}\left[1 - \frac{3}{2}\int_0^1 (1 - y)^{1/2}\cos\bar{\psi} y dy\right] \quad (3)$$

A graph of this equation produces a curve in d, $\bar{\psi}$ space. The region above this curve is linearly unstable, the region below the curve linearly stable. Practical values of $\bar{B}$ and $\bar{\tau}$ cause d to fall somewhat below the lowest point on such a curve. This indicates that the combustion model adopted does not provide a strong enough response to pressure oscillations to drive on instability. In consequence of this, solutions of Eq. (2) for large amplitude waves probably have little practical significance. The conclusion reached is that the combustion model employed is not sufficiently realistic. It is probable that a more sophisticated combustion model would provide a governing equation for f possessing greater practical significance. It is suggested that the use of the multiple scale technique demonstrated here will prove useful in such an analysis (as it certainly was in the present work) in the reduction of the governing system of nonlinear partial differential equations to a single ordinary differential equation determining the nonlinear behavior of the oscillations.

References

[1] Mitchell, C. E., Crocco, L., and Sirignano, W. A., "Nonlinear Longitudinal Instability in Rocket Motors with Concentrated Combustion," *Combustion Science and Technology,* Vol. 1, 1969, pp. 35–64.

[2] Crocco, L. and Mitchell, C. E., "Nonlinear Periodic Oscillations in Rocket Motors with Distributed Combustion," *Combustion Science and Technology,* Vol. 1, 1969, pp. 147–169.

[3] Crocco, L. and Cheng, S. I., *Theory of Combustion Instability in Liquid Propellant Rocket Motors,* Butterworths, London, 1956, pp. 12–24.

[4] Priem, R. J. and Guentert, D. C., "Combustion Instability Limits Determined by a Nonlinear Theory and a One Dimensional Model," NASA TN D-1409, 1962, NASA.

[5] Cole, J. D., *Perturbation Methods in Applied Mathematics,* Blaisdell Publishing Co., Waltham, Mass., 1968.

Numerical Solution of Boundary-Layer Flows with Massive Blowing

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Introduction

DESCRIPTIONS of laminar boundary layers with large mass injection rates are required in several applications. Specifically, of current interest is the stagnation region on the heat shield for a Jupiter atmospheric entry probe.[1] In this case, the mass transfer from the wall due to radiative heating is so great that any convective heating is blocked.

The structure of boundary layers with massive blowing consists of a thin region with large gradients that separates the outer stream from an inner layer adjacent to the wall where gradients are either absent or very small. This boundary-layer structure has previously been examined by several investigators.[2–4]

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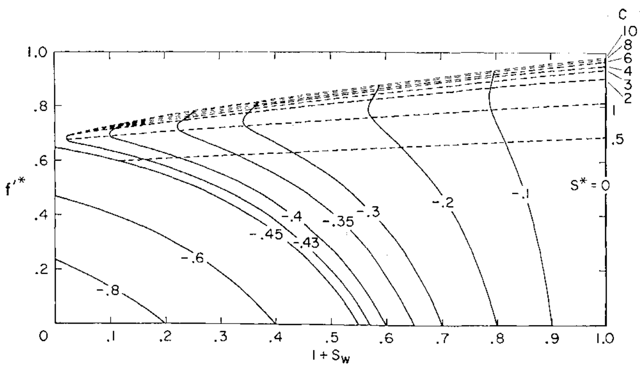

Fig. 1 Dividing streamline velocity.

The usual numerical methods for handling the two-point boundary conditions, which have successfully been applied to boundary layers without mass transfer, break down when applied to cases with mass transfer when the blowing parameter is large. This failure has been attributed to the lack of convergence[5] and the instabilities[4] of the usual numerical methods. In any case, inordinately precise wall values are required for solving the two-point boundary-value problem, and several analytic methods have been proposed to treat the problem approximately.[3–6] However, these approximate methods cannot be generalized to treat accurately hypersonic boundary layers with chemical reactions and multicomponent transport phenomena.

This Note describes a numerical method that avoids the numerical difficulties previously mentioned; the results of calculations obtained by means of the new method are given.

Method Description

For boundary layers in stagnation regions with massive blowing, all gradients adjacent to the wall tend to vanish as the blowing increases. With this boundary-layer structure, the usual procedure of specifying gradients at the wall and subsequently modifying them (to satisfy the outer boundary conditions) clearly cannot work. For the case of blowing into boundary layers in stagnation regions, there is a dividing streamline where the oncoming flow is stopped by the injected flow; in the case of massive blowing, the dividing streamline is embedded in the thin region with large gradients.

The present method incorporates the dividing streamline as the origin of coordinates, retaining the similarity variable η as the independent variable, as opposed to introducing the nondimensional stream function f as a new independent variable as in Ref. 4. With no blowing, of course, the dividing streamline coincides with the wall, and the coordinate system introduced herein reduces to the usual ones. The simple strategem previously described reduces by orders of magnitude the precision required for the specification of the starting values to solve the boundary-value problem.

The new choice of coordinates essentially converts the two-point boundary-value problem into a three-point boundary-value problem; the additional boundary condition thus introduced is readily satisfied, as outlined below. The proposed method incorporates the automated procedure of Ref. 7 for satisfying the asymptotic boundary conditions at the outer edge of the boundary layer, with modifications to accommodate massive blowing.

Application of the method can be illustrated as follows. Consider the similar boundary-layer equations with unit Prandtl number and with pressure gradient parameter $\beta = \frac{1}{2}$:

$$f''' + ff'' + \tfrac{1}{2}(S + 1 - f'^2) = 0 \qquad (1)$$

$$S'' + fS' = 0 \qquad (2)$$

$$f'(\eta_w) = 0 \qquad f(\eta_w) \equiv -C \qquad S(\eta_w) = S_w \qquad (3)$$

$$f'(\infty) = 1 \qquad S(\infty) = 0 \qquad (4)$$

In these equations, the independent variable η is zero when f is zero. With blowing (C different from zero) η_w will always be negative. The subscript w denotes values at the wall, and the notation is that of Ref. 8.

The method consists of assigning at $\eta = 0$, the five quantities $f(0) = 0$, $f'(0) \equiv f'^*$, $f''(0)$, $S(0) \equiv S^*$, and $S'(0)$. The specification $f(0) = 0$ merely locates the origin of coordinates at the dividing streamline. The two quantities f'^* and S^* are merely parameters that characterize the solution. They are arbitrarily assigned numerical values, as are the missing initial conditions $f''(0)$ and $S'(0)$ subject to the modifications outlined below.

The procedure of Ref. 7 searches for the edge of the boundary layer and modifies the missing initial conditions $f''(0)$ and $S'(0)$ so as to satisfy the boundary conditions at the outer edge of the boundary layer. When the searching procedure is applied to the case of massive blowing, the greater the value of the blowing parameter C, experience shows the more rapid the convergence of the procedure.

A remaining boundary condition at the wall, $f' = 0$ (the third of the three-point boundary conditions), is readily satisfied by integrating backward from $\eta = 0$ with the starting values determined by the searching method and stopping the integration when $f' = 0$. At this point, the values of f_w and S_w are accepted at their current values, and the solution of a boundary-value problem is obtained. Note that f_w and S_w are not preassigned; rather, f'^* and S^* are preassigned. Specification of values at the dividing streamline may appear to be an artificial procedure; once the solutions are available, however, this objection can be overcome by relating the wall values to the values at the dividing streamline as will be done for the numerical example treated below.

Numerical Results

The method previously described was programed in FORTRAN IV, double precision, on the IBM-7094 located at Ames Research Center. The procedure used to obtain solutions with large values of C, the blowing parameter, was to fix S^* and to increment f'^* from an initial value of zero. Of course, with $f'^* = 0$ the solutions obtained herein could be compared with the solutions available for zero blowing. Where comparisons could be made with the results of Refs. 8 and 9, the solutions coincided in all cases.

Figure 1 presents the variation of the velocity on the dividing streamline (dashed curves) with C and $1 + S_w$, the wall enthalpy. The dashed curve $C = 2$ compares favorably with the analytic result given in Fig. 10 of Ref. 4. The solid curves indicate where the enthalpy function S^* is constant on the dividing streamline. The region above the knees in these curves corresponds to massive blowing and the knees themselves to the onset of the rapid diminishing of gradients at the wall.

Figure 2 shows two sets of boundary-layer profiles for approximately the same value of wall enthalpy $1 + S_w$ corresponding to two different blowing rates, one for a value of f'^* below the knee of the solid curve marked $S^* = -0.2$ in Fig. 1 and the other for a value of f'^* above the knee. It can be seen from the figure that there is no heat transfer to the wall, and the velocity profile displays an inflexion point when the blowing parameter is large. Consequently, in this case, the maximum shear does not occur at the wall as it does when there is no heat transfer.[3]

Concluding Remarks

The calculations performed using the proposed direct numerical method indicate that it is possible to integrate the boundary-layer equations under conditions of massive blowing

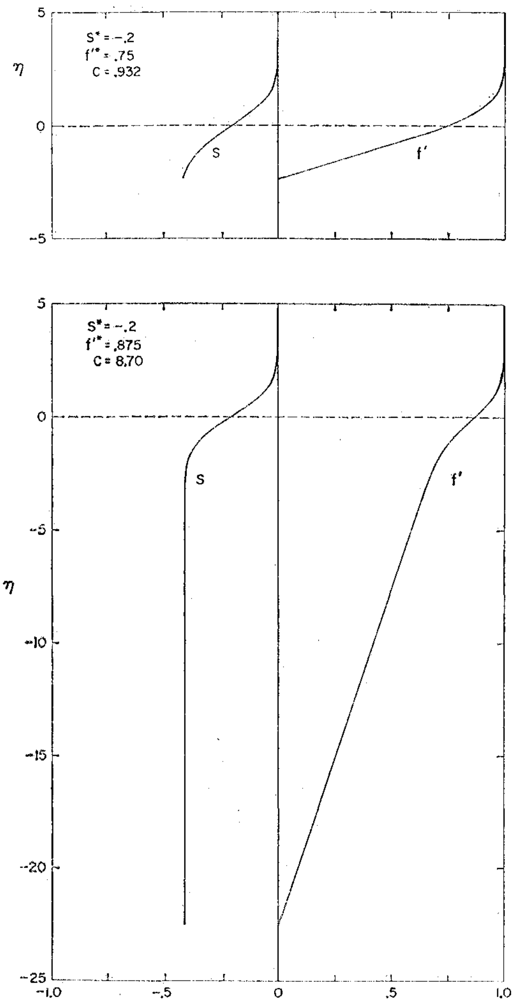

Fig. 2 Boundary-layer profiles.

irrespective of the value of the blowing parameter. Unlike the usual numerical methods, which cease converging to the solution when the blowing parameter increases, the proposed method actually converges to the solution more rapidly as the value of the blowing parameter increases.

The satisfaction of the two-point boundary conditions is a problem common to all means of solving the boundary-layer equations; the method proposed herein should aid in improving the schemes for solving complex boundary-layer problems.

References

[1] Tauber, M., "Atmospheric Entry Into Jupiter," *Journal of Spacecraft and Rockets*, Vol. 6, No. 10, Oct. 1969, pp. 1103–1109.

[2] Libby, P. A., "The Homogeneous Boundary Layer at an Axisymmetric Stagnation Point with Large Rates of Injection," *Journal of the Aerospace Sciences*, Vol. 29, No. 1, Jan. 1962, pp. 48–60.

[3] Aroesty, J. and Cole, J. D., "Boundary Layer Flows with Large Injection Rates," Memo-RM-4620-ARPA, 1965, Rand Corporation, Santa Monica, Calif.

[4] Kubota, T. and Fernandez, F. L., "Boundary Layer Flows with Large Injection and Heat Transfer," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 22–28.

[5] Libby, P. A., "On the Numerical Analysis of Stagnation Point Flows with Massive Blowing," *AIAA Journal*, Vol. 8, No. 11, Nov. 1970, pp. 2095–2096.

[6] Pretsch, J., "Analytic Solutions of the Laminar Boundary Layer with Asymptotic Suction and Injection," *Zeitschrift für angewandt Mathematik und Mechanic*, Vol. 24, No. 5, June 1944, pp. 264–267.

[7] Nachtsheim, P. R. and Swigert, P., "Satisfaction of Asymptotic Boundary Conditions in the Numerical Solution of Boundary Layer Equations," *Proceedings of the Ninth Midwestern Mechanics Conference*, The University of Wisconsin, Madison, Aug. 1965, pp. 361–371.

[8] Cohen, C. B. and Reshotko, E., "Similar Solutions for the Compressible Laminar Boundary Layer with Heat Transfer and Pressure Gradient," Rept. 1293, 1956, NACA.

[9] Christian, J. W., Hankey, W. L., and Petty, J. S., "Similar Solutions of the Attached and Separated Compressible Laminar Boundary Layer with Heat Transfer and Pressure Gradient," Rept. 70-0023, Feb. 1970, Aerospace Research Labs., Air Force Systems Command, Wright-Patterson Air Force Base, Ohio.

Hypersonic Strong Interaction Flow over an Inclined Surface

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THE purpose of this Note is to present a solution for the hypersonic strong interaction flow over an inclined surface. An asympotic expansion in powers of the hypersonic interaction parameter $\bar{\chi}$ is used to reduce the boundary-layer equations to a sequence of ordinary differential equations. The scheme was originally suggested in Refs. 1 and 2. References 3 and 4 give the zero-order solutions for $Pr = 1$. In Ref. 5 a two-term solution, which corresponds to the zeroth and the second-order solutions of the present scheme, was obtained by using the Karman-Pohlhausen integral method for the flow over an insulated flat plate. Here some results are presented for $Pr = 1$ and 0.72 with different thermal conditions on the plate.

Let the physical variables be denoted by the superscript* and let the subscripts ∞ and e represent the conditions in the freestream and in the inviscid outer flow (Fig. 1). Theconditions in the boundary layer are represented without any subscript. The dependent variables are nondimensionalized with respect to their freestream values, and the independent variables x^*,y^* with respect to a characteristic length dimension L. The nondimensionalized quantities are represented without any superscript. The gas is assumed to have constant c_p,γ and Pr and obey a linear viscosity-temperature relation $\mu = CT$, C being a constant.

The pressure distribution on the surface and the boundary-layer displacement thickness are given by[1,2]:

$$p(x) = \frac{p^*(x^*)}{p_\infty^*} = p_0\bar{\chi}\left[1 + \frac{p_1K_b}{\bar{\chi}^{1/2}} + \frac{p_2 + p_3K_b^2}{\bar{\chi}} + 0(\bar{\chi}^{-3/2})\right] \tag{1}$$

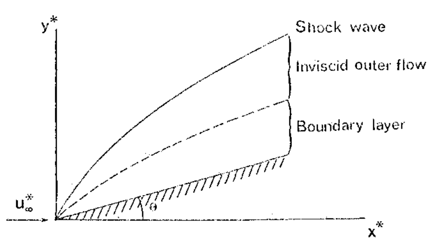

Fig. 1 Flowfield above an inclined plate in hypersonic viscous flow.

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